Problem A.5

Prove the Schwarz inequality (Equation A.27). *Hint:* Let $|\gamma\rangle = |\beta\rangle - (\langle\alpha\,|\,\beta\rangle\,/\,\langle\alpha\,|\,\alpha\rangle)\,|\alpha\rangle$, and use $\langle\gamma\,|\,\gamma\rangle \geq 0$.

Solution

Here the aim is to prove the Schwarz inequality.

$$|\langle\alpha\,|\,\beta\rangle|^2 \leq \langle\alpha\,|\,\alpha\rangle\langle\beta\,|\,\beta\rangle \tag{A.27}$$

Consider the vector,

$$|\gamma\rangle = |\beta\rangle - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\,|\alpha\rangle,$$

and the inner product with itself.

$$\begin{aligned}
\langle\gamma\,|\,\gamma\rangle &= \langle\gamma|\left(|\beta\rangle - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\,|\alpha\rangle\right) \\
&= \langle\gamma\,|\,\beta\rangle - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\,\langle\gamma\,|\,\alpha\rangle \\
&= \langle\beta\,|\,\gamma\rangle^* - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\,\langle\alpha\,|\,\gamma\rangle^* \\
&= \left[\langle\beta|\left(|\beta\rangle - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\,|\alpha\rangle\right)\right]^* - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\left[\langle\alpha|\left(|\beta\rangle - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\,|\alpha\rangle\right)\right]^* \\
&= \left(\langle\beta\,|\,\beta\rangle - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\,\langle\beta\,|\alpha\rangle\right)^* - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\left(\langle\alpha\,|\,\beta\rangle - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\,\langle\alpha\,|\alpha\rangle\right)^* \\
&= \left(\langle\beta\,|\,\beta\rangle - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\,\langle\alpha\,|\beta\rangle^*\right)^* - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}\left(\langle\alpha\,|\,\beta\rangle - \langle\alpha\,|\,\beta\rangle\right)^* \\
&= \left(\langle\beta\,|\,\beta\rangle - \frac{|\langle\alpha\,|\,\beta\rangle|^2}{\langle\alpha\,|\,\alpha\rangle}\right)^* - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}(0)^* \\
&= \left(\langle\beta\,|\,\beta\rangle - \frac{|\langle\alpha\,|\,\beta\rangle|^2}{\langle\alpha\,|\,\alpha\rangle}\right) - \frac{\langle\alpha\,|\,\beta\rangle}{\langle\alpha\,|\,\alpha\rangle}(0) \\
&= \langle\beta\,|\,\beta\rangle - \frac{|\langle\alpha\,|\,\beta\rangle|^2}{\langle\alpha\,|\,\alpha\rangle}
\end{aligned}$$

Since $\langle\gamma\,|\,\gamma\rangle \geq 0$,

$$\langle\beta\,|\,\beta\rangle - \frac{|\langle\alpha\,|\,\beta\rangle|^2}{\langle\alpha\,|\,\alpha\rangle} \geq 0 \quad \rightarrow \quad -\frac{|\langle\alpha\,|\,\beta\rangle|^2}{\langle\alpha\,|\,\alpha\rangle} \geq -\langle\beta\,|\,\beta\rangle.$$

Therefore, multiplying both sides by $-\langle\alpha\,|\,\alpha\rangle$,

$$|\langle\alpha\,|\,\beta\rangle|^2 \leq \langle\alpha\,|\,\alpha\rangle\langle\beta\,|\,\beta\rangle.$$